Homework Problems V
PHYS 425: Electromagnetism I

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1. Griffiths Problem 5.30

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{r}')}{\mathscr{r}} dV'$$

Because the derivatives only operate on $\boldsymbol{r}$ in $\mathscr{r}$,

$$\nabla \cdot \boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \nabla \cdot \left(\frac{\boldsymbol{J}(\boldsymbol{r}')}{\mathscr{r}}\right) dV' = 0 + \frac{\mu_0}{4\pi}\int \boldsymbol{J}(\boldsymbol{r}') \cdot \nabla\left(\frac{1}{\mathscr{r}}\right) dV'$$

$$= -\frac{\mu_0}{4\pi}\int \boldsymbol{J}(\boldsymbol{r}') \cdot \nabla'\left(\frac{1}{\mathscr{r}}\right) dV',$$

where ∇' means the gradient with respect to $\boldsymbol{r}'$. Now use the integration by parts formula

$$\nabla \cdot \boldsymbol{A}(\boldsymbol{r}) = -\frac{\mu_0}{4\pi}\int \nabla' \cdot \left(\frac{\boldsymbol{J}(\boldsymbol{r}')}{\mathscr{r}}\right) dV' + \frac{\mu_0}{4\pi}\int \frac{\nabla' \cdot \boldsymbol{J}(\boldsymbol{r}')}{\mathscr{r}} dV'.$$

By the divergence theorem, the first integral becomes a surface integral. Letting the surface go to infinity yields zero because $\boldsymbol{J}$ vanishes there. For steady currents $\nabla \cdot \boldsymbol{J} = 0$ and the second term vanishes. Thus $\nabla \cdot \boldsymbol{A} = 0$.

Because the derivatives only operate on $\boldsymbol{r}$ in $\mathscr{r}$,

$$\nabla \times \boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \nabla \times \left(\frac{\boldsymbol{J}(\boldsymbol{r}')}{\mathscr{r}}\right) dV' = 0 - \frac{\mu_0}{4\pi}\int \boldsymbol{J}(\boldsymbol{r}') \times \nabla\left(\frac{1}{\mathscr{r}}\right) dV'$$

$$= \frac{\mu_0}{4\pi}\int \boldsymbol{J}(\boldsymbol{r}') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} dV' = \boldsymbol{B}(\boldsymbol{r})$$

$$\nabla^2 \boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \nabla^2 \left(\frac{\boldsymbol{J}(\boldsymbol{r}')}{\mathscr{r}}\right) dV' = 0 + \frac{\mu_0}{4\pi}\int \boldsymbol{J}(\boldsymbol{r}') \nabla^2\left(\frac{1}{\mathscr{r}}\right) dV'$$

$$= \frac{\mu_0}{4\pi}\int \boldsymbol{J}(\boldsymbol{r}')\left(-4\pi\delta^3(\boldsymbol{r} - \boldsymbol{r}')\right) dV' = -\mu_0 \boldsymbol{J}(\boldsymbol{r})$$

2. Griffiths Problem 6.7

As there are no free currents, $\boldsymbol{H}$ vanishes. So inside the cylinder

$$\boldsymbol{B} = \mu_0 \boldsymbol{M}$$

Outside the cylinder

$$\boldsymbol{B} = \mu_0 (\boldsymbol{H} + \boldsymbol{M}) = 0 + 0 = 0$$

Note the bound surface current $\boldsymbol{K}_b = \boldsymbol{M} \times \hat{n} = M\hat{\phi}$ is just that required to cancel the inside field via Ampere's law.

$$\oint B \cdot dl = B_+ dl - B_- dl = -\mu_0 K dl$$
$$B_+ = 0 \to B_- = \mu_0 M$$

This same surface current is important for Problem 6.16

3. Griffiths Problem 6.13

By superposition, the field is the same as that of a uniform magnetized material, plus a sphere magnetized in the opposite direction. By example 6.1, inside a uniformly magnetized sphere

$$\boldsymbol{B} = \frac{2}{3}\mu_0 \boldsymbol{M}$$

When subtracted from the initial field

$$\boldsymbol{B} = \boldsymbol{B}_0 - \frac{2}{3}\mu_0 \boldsymbol{M}.$$

Because ***M*** vanishes inside the sphere

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - 0 = \frac{\boldsymbol{B}_0}{\mu_0} - \frac{2}{3}\boldsymbol{M} = \boldsymbol{H}_0 + \frac{\boldsymbol{M}}{3}$$

For the needle, because there is no free current and ***H*** is aligned along the needle, ***H*** is the same inside and outside. So

$$B_{inside} = \mu_0 H_{inside} = \mu_0 H_{outside} = \mu_0 \left[\frac{\boldsymbol{B}_0}{\mu_0} - \boldsymbol{M} \right] = \boldsymbol{B}_0 - \mu_0 \boldsymbol{M}$$

For the wafer, ***B*** is normal to the wafer surface and hence the same inside and outside. So

$$\boldsymbol{H}_{inside} = \boldsymbol{B}_{inside} / \mu_0 = \boldsymbol{B}_{outside} / \mu_0 = \boldsymbol{H}_{outside} + \boldsymbol{M}$$

4. Griffiths Problem 6.16

By the right hand rule and Ampere's Law

$$\boldsymbol{H} = H_\phi \hat{\phi}$$
$$2\pi s H_\phi = I$$
$$H_\phi = \frac{I}{2\pi s}$$

where $\hat{\phi}$ points out above the central conductor and in below the central conductor in Fig. 6.24. For a linear material with susceptability χ_m

$$\boldsymbol{B} = \mu \boldsymbol{H} = \mu_0 (1 + \chi_m) \boldsymbol{H}.$$

The magnetization is $\boldsymbol{M} = \chi_m \boldsymbol{H} = \chi_m I/(2\pi s)$. The bound current is

$$\boldsymbol{J}_b = \nabla \times \boldsymbol{M} = \left[\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\chi_m I}{2\pi s}\right)\right]\hat{z} = 0.$$

The bound surface current at $s = a$ is

$$\boldsymbol{K}_b = \boldsymbol{M} \times \hat{n} = -\boldsymbol{M} \times \hat{s} = \frac{\chi_m I}{2\pi a}\hat{z}$$

Ampere's Law gives

$$2\pi s B_\phi = \mu_0\left(I + 0 + \frac{\chi_m I}{2\pi a}2\pi a\right)$$

$$B_\phi = \frac{\mu_0(1+\chi_m)I}{2\pi s},$$

as above.

5. Griffiths Problem 6.21 Hint: do a line integral using Equation 6.3

a) In a region without magnetic field at large radius, rotate the dipole into final orientation. Because the field vanishes there is no force and no work to orient the dipole. Next integrate Eq. 6.3.

$$U = -\int \boldsymbol{F}\cdot d\boldsymbol{l} = -\int_{\infty}^{r} \nabla(\boldsymbol{m}\cdot\boldsymbol{B})\cdot d\boldsymbol{l}$$

$$= -\boldsymbol{m}\cdot\boldsymbol{B}(\boldsymbol{r})$$

b) Putting a dipole $\boldsymbol{m}_1$ into the field of $\boldsymbol{m}_2$ (the answer is symmetric in the choice!)

$$U = -\boldsymbol{m}_1 \cdot \left\{\frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\boldsymbol{m}_2\cdot\hat{r})\hat{r} - \boldsymbol{m}_2\right]\right\}$$

$$= \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[\boldsymbol{m}_1\cdot\boldsymbol{m}_2 - 3(\boldsymbol{m}_1\cdot\hat{r})(\boldsymbol{m}_2\cdot\hat{r})\right]$$

c) Using the geometry given in the problem

$\boldsymbol{m}_1 = \cos\theta_1\hat{x} + \sin\theta_1\hat{y} \quad \boldsymbol{m}_2 = \cos\theta_2\hat{x} + \sin\theta_2\hat{y} \quad \hat{r} = \hat{x}$

$$U = \frac{\mu_0}{4\pi}\frac{m_1 m_2}{r^3}\left[\cos(\theta_1 - \theta_2) - 3\cos\theta_1\cos\theta_2\right] = \frac{\mu_0}{4\pi}\frac{m_1 m_2}{r^3}\left[\sin\theta_1\sin\theta_2 - 2\cos\theta_1\cos\theta_2\right]$$

$$\frac{\partial U}{\partial\theta_1} = \cos\theta_1\sin\theta_2 + 2\sin\theta_1\cos\theta_2 = 0 \quad \frac{\partial U}{\partial\theta_2} = \sin\theta_1\cos\theta_2 + 2\cos\theta_1\sin\theta_2 = 0$$

$\theta_1 = \theta_2 = 0; \quad \theta_1 = \pi/2, \theta_2 = \pm\pi/2$

d) Minimum energy in first case when the moments line up.

6. Griffiths Problem 6.24 Hint: take a gradient of the answer to (b) in the previous problem, or use Equation 6.3 directly to get the magnetic forces.

a) For the dipoles anti-aligned and oriented in the vertical direction the mechanical potential is

$$U = \frac{\mu_0}{4\pi}\frac{m^2}{r^3}[3-1]$$

where r is the height difference between the dipoles. The dipole force is up

$$-\frac{dU}{dr}=\frac{3\mu_0}{2\pi}\frac{m^2}{r^4}$$

and balances the weight $m_d g$. The height of the levitated dipole is

$$\frac{3\mu_0}{2\pi}\frac{m^2}{m_d g}=r^4 \rightarrow r=\left(\frac{3\mu_0}{2\pi}\frac{m^2}{m_d g}\right)^{1/4}$$

b) Using the distances given, the distances between the dipoles are x, y, and $x+y$. The potential of all three dipole interactions plus the gravitational interaction is

$$U=\frac{2\mu_0}{4\pi}\frac{m^2}{x^3}+\frac{2\mu_0}{4\pi}\frac{m^2}{y^3}-\frac{2\mu_0}{4\pi}\frac{m^2}{(x+y)^3}+m_d gx+m_d g(x+y)$$

The equilibria solve

$$\frac{\partial U}{\partial x}=0\rightarrow\frac{3\mu_0}{2\pi}\frac{m^2}{x^4}-\frac{3\mu_0}{2\pi}\frac{m^2}{(x+y)^4}=2m_d g$$
$$\frac{\partial U}{\partial y}=0\rightarrow\frac{3\mu_0}{2\pi}\frac{m^2}{y^4}-\frac{3\mu_0}{2\pi}\frac{m^2}{(x+y)^4}=m_d g$$

These are non-linear equations that must be solved numerically. Here is one method. Take twice the second equation and subtract it from the first. The result is

$$\frac{1}{x^4}-\frac{2}{y^4}+\frac{1}{(x+y)^4}=0\rightarrow\frac{y^4}{x^4}-2+\frac{y^4}{x^4\left(1+{}^{y}\!/\!_{x}\right)^4}=0$$

Letting $z=(y/x)^4$

$$z=2+\frac{z}{\left(1+z^{1/4}\right)^4}=0.$$

Iterate numerically starting with $z=2$ in the right hand side to compute the left hand side. I get $z_0=2$, $z_1=1.912927$, $z_2=1.914684$, $z_3=1.914648$, and $z_4=1.914649$ after which the value doesn't change to the digits indicated. Thus $x/y=(1.914649)^{-1/4}=0.8501$.